Stochastic Lorenz model for periodically driven Rayleigh-Bénard convection

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The order-disorder transition observed in periodically driven Rayleigh-Bénard convection is studied by extending the generalized Lorenz model introduced by Ahlers, Hohenberg, and Lücke [Phys. Rev. A **32**, 3493 (1985)] to include the effects of thermal noise. It is shown that this stochastic Lorenz model predicts, for thermal noise intensities, an order-disorder transition line much closer to the experimental values than the prediction of previous models. This result makes clear that a dynamical description allowing for inertial effects is needed to account for the behavior of systems dynamically forced to cross an instability threshold. [S1063-651X(97)50304-7]

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The subject of externally modulated Rayleigh-Bénard convection has received considerable theoretical [1–4] and experimental [5–8] attention. On the one hand, Rayleigh-Bénard convection has become the paradigm of pattern-forming systems. On the other, it is hoped that the dynamic modulation of the system through its pattern-forming instability can clarify the role of fluctuations in the early stages of pattern formation.

The influence of noise on the onset of Rayleigh-Bénard convection has been extensively studied [6,7,9–12]. Noise is generally assumed to be the mechanism responsible of destabilizing the conductive state above the convective threshold, so its intensity is overly important in determining the dynamics of convective pattern formation. The usual theoretical approach to the problem is the reduction of the Navier-Stokes equations to a number of model equations, which are supposed to embrace the main features of the system [8]. The archetype of these is the Swift-Hohenberg (SH) equation, which is a nonlinear partial differential equation for an order parameter [10]. The intensity of the (internal) thermal noise is most usually computed through fluctuation-dissipation arguments, when the system is below the convective threshold and in a stationary state [10]. For the fluid in a Rayleigh-Bénard cell, the thermal noise strength can be shown to be $F_{\rm th}\sim k_BT/(\rho\ d\nu^2)$ for both free-slip [9] and rigid [3] vertical boundary conditions, where ρ is the mass density, ν the kinematic viscosity, and d the plate separation. Below the convective threshold, recent experiments by Wu, Ahlers, and Cannell [12] verified the predictions of these models.

Above threshold, the noise intensity is argued to be the same even for systems out of equilibrium, which is justified by linearizing the Navier-Stokes equations around the adequate conductive state [8,10], although this approach has been questioned in the frame of generalized fluctuations away from equilibrium [13]. But in experiments that periodically modulate the control parameter through the convective threshold, such as those performed several years ago by Meyer, Ahlers, and Cannell [6], a much greater (typically by a factor $\sim 10^4$) noise strength is needed to account for the observed order-disorder transition, and there is not yet any explanation for this discrepancy. A similar disagreement was observed in the experiments of Ref. [7], where the heat current is linearly ramped through the threshold. Despite a great deal of work by many authors, a mechanism enabling thermal noise alone to provide the driving force for the onset of convection in these *dynamic* experiments has not yet been proposed [3,8,7].

In a recent paper we investigated the possibility that the results of Meyer, Ahlers, and Cannell [6] can be accounted for by the inclusion of *external* noise sources [4]. In that work we modeled the external noise as suggested by García-Ojalvo, Hernández-Machado, and Sancho [14] describing temperature fluctuations in the plates of the cell as noise in the control parameter of the corresponding SH equation. We showed that external noise intensities compatible with recent experiments [12] are far too weak to reproduce the results of Ref. [6]. This negative result makes it unplausible that the inclusion of external noise in the usual model equations can solve the problem.

It is currently well understood that the noise effect is decisive at times near the crossing of the convective threshold [8]; hence a detailed description of the dynamics is crucial. Several years ago Ahlers, Hohenberg, and Lücke [1] introduced a generalized Lorenz model description of periodically driven Rayleigh-Bénard convection, showing that the system dynamics near the convective threshold is very different from that in static settings. The model equations of that work reduce to the usual amplitude equation [11] in adequate limits, but they cannot be obtained by simply replacing a time-periodic driving term in the model equations derived for static forcing. This leads to questioning if the inclusion of thermal noise in this model can give a more precise description of the experimental results of Ref. [6].

In this work we will formulate a generalized Lorenz model like the one in Ref. [1], including projections of the thermal noise fields on the relevant hydrodynamic modes. We will use the model to predict the order-disorder transition observed in the experiments of Ref. [6]. We will show that, though this model still requires a greater-than-thermal noise intensity to fit the experimental results, the prediction for the thermal noise intensity is much closer to the experiments than that obtained from the corresponding SH equation. This shows that the importance of an accurate description of the dynamical effects produced by a periodic driving cannot be neglected, and that these effects seem not to be embraced by the usual stochastic models.

We consider a fluid layer between two laterally infinite horizontal plates separated by a distance d. The density of the fluid is ρ, its temperature is T, its velocity is $\boldsymbol{u}$, and its pressure is p. The kinematic viscosity is denoted by ν, the thermal diffusivity by κ, and the gravitational acceleration by g. Introducing dimensionless variables by the scaling $t\to(\kappa/d^2)t$, $l\to l/d$, $\Delta T\to R$, where $R=(\alpha g d^3\Delta T)/(\kappa\nu)$ is the Rayleigh number, and defining the Prandtl number $\sigma=\nu/\kappa$, the Oberbeck-Boussinesq equations read [1]

$$(\partial_t+\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}=\sigma\nabla^2\mathbf{u}+\sigma T\hat{z}-\boldsymbol{\nabla}p+\boldsymbol{\nabla}\cdot\mathbf{s}, \tag{1a}$$

$$(\partial_t+\mathbf{u}\cdot\boldsymbol{\nabla})T=\nabla^2 T-\boldsymbol{\nabla}\cdot\mathbf{q}, \tag{1b}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}=0. \tag{1c}$$

Thermal noise is modeled by the zero-mean Gaussian white noise fields $\mathbf{s}$ and $\mathbf{q}$, with self-correlations [8]

$$\langle q_i(\mathbf{r},t)q_j(\mathbf{r},t)\rangle=c_q\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\delta_{ij}, \tag{2a}$$

$$\langle s_{ij}(\mathbf{r},t)s_{lm}(\mathbf{r}',t')\rangle=c_s\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl}), \tag{2b}$$

and thermodynamic intensities

$$c_s=\left(\frac{k_{\rm B}T}{\rho d\nu^2}\right)2\sigma^3,\quad c_q=\frac{2(Td^3\alpha g/\kappa\nu)^2}{(c_{\rm V}d^3/k_{\rm B})}. \tag{3}$$

The dimensionless temperature $T=T^{\rm c}+\theta$ is expressed in terms of the conduction profile $T^{\rm c}(z,t)$ satisfying the heat conduction equation

$$\partial_t T^{\rm c}=\partial_z^2 T^{\rm c}. \tag{4}$$

The periodic driving is accounted for by the boundary conditions, which we will take to be the same as in the experimental setting of Ref. [6],

$$T^{\rm c}(0,t)=T^l(t)=T_l+A\cos(\omega t), \tag{5a}$$

$$T^{\rm c}(1,t)=T^u(t)=T_u. \tag{5b}$$

Working for simplicity with free-slip boundary conditions, the Lorenz model is obtained by introducing the expansions

$$u_j(\mathbf{r},t)=i\sum_{\mathbf{n}}\tilde{u}_j(\mathbf{n},t)e^{i\mathbf{q}(\mathbf{n})\cdot\mathbf{r}},\quad j=x,y,z, \tag{6a}$$

$$\theta(\mathbf{r},t)=i\sum_{\mathbf{n}}\tilde{\theta}(\mathbf{n},t)e^{i\mathbf{q}(\mathbf{n})\cdot\mathbf{r}}, \tag{6b}$$

with

$$\mathbf{n}=(n_x,n_y,n_z),\quad n_i=0,\pm1,\pm2,\ldots, \tag{7a}$$

$$\mathbf{q}(\mathbf{n})=(k_x n_x,k_y n_y,k_z n_z). \tag{7b}$$

The fields $s_{ij}(\mathbf{r},t)$ and $q_i(\mathbf{r},t)$ are similarly expanded. As in Ref. [1] we take $k_z=\pi$ and $k_x=k_y=k_{\rm c}$, the critical wave number for horizontal rolls, corresponding to the critical Rayleigh number $R=R_{\rm c}$ in the absence of modulation. Replacing expansions (6) in Eqs. (1), and keeping only the lowest modes giving a nontrivial truncation (corresponding to straight rolls along the x direction), we obtain

$$\partial_t\tilde{u}_x(\mathbf{n}_1,t)=-\sigma(k_{\rm c}^2+\pi^2)\tilde{u}_x(\mathbf{n}_1,t)-\sigma\frac{\pi k_{\rm c}}{k_{\rm c}^2+\pi^2}\tilde{\theta}(\mathbf{n}_1,t)+\xi_1(t), \tag{8a}$$

$$\partial_t\tilde{\theta}(\mathbf{n}_1,t)=-(k_{\rm c}^2+\pi^2)\tilde{\theta}(\mathbf{n}_1,t)-\frac{k_{\rm c}}{\pi}\tilde{u}_x(\mathbf{n}_1,t)\times[R(t)-2\pi\tilde{S}^\infty(2,t)-2\pi\tilde{\theta}(\mathbf{n}_2,t)]+\xi_2(t), \tag{8b}$$

$$\partial_t\tilde{\theta}(\mathbf{n}_2,t)=-(2\pi)^2\tilde{\theta}(\mathbf{n}_2,t)-4k_{\rm c}\tilde{u}_x(\mathbf{n}_1,t)\tilde{\theta}(\mathbf{n}_1,t)+\xi_3(t), \tag{8c}$$

with

$$R(t)=T^l(t)-T^u(t), \tag{9}$$

$$T^{\rm c}(z,t)=R(t)(1-z)+S^\infty(z,t)+T^u(t), \tag{10}$$

$$S^\infty(z,t)=\sum_{n=1}^{\infty}\tilde{S}^\infty(n,t)\sin(n\pi z), \tag{11}$$

and $\mathbf{n}_1=(1,0,1)$ and $\mathbf{n}_2=(0,0,2)$. Here $S^\infty(z,t)$ is the deviation of the conductive temperature profile from the linear profile corresponding to the instantaneous Rayleigh number $R(t)$. The superscript indicates that it corresponds to a laterally infinite system, and is kept for consistency with the notation of Ref. [1]. The noises are given by

$$\xi_1(t)=\frac{i}{k_{\rm c}^2+\pi^2}[\pi^2k_{\rm c}\tilde{s}_{xx}(\mathbf{n}_1,t)+\pi^3\tilde{s}_{xz}(\mathbf{n}_1,t)-\pi k_{\rm c}^2\tilde{s}_{zx}(\mathbf{n}_1,t)-\pi^2k_{\rm c}\tilde{s}_{zz}(\mathbf{n}_1,t)], \tag{12a}$$

$$\xi_2(t)=-i[k_{\rm c}\tilde{q}_x(\mathbf{n}_1,t)+\pi\tilde{q}_z(\mathbf{n}_1,t)], \tag{12b}$$

$$\xi_3(t)=-i2\pi\tilde{q}_z(\mathbf{n}_2,t). \tag{12c}$$

In deriving these equations, symmetries imposed by the boundary conditions have been used. Using these symmetries and Eqs. (2), it is straightforward to obtain

$$\langle\tilde{q}_i^*(\mathbf{n},t)\tilde{q}_j(\mathbf{n}',t')\rangle=\frac{k_{\rm c}^2}{2(2\pi)^2}c_q\delta_{ij}\delta(t-t')\times[\delta_{\mathbf{n},\mathbf{n}'}+(\delta_{iz}-\delta_{ix}-\delta_{iy})\delta_{\mathbf{n},\tilde{\mathbf{n}}'}], \tag{13a}$$

$$\langle\tilde{s}_{ij}^*(\mathbf{n},t)\tilde{s}_{lm}(\mathbf{n}',t')\rangle=\frac{k_{\rm c}^2}{2(2\pi)^2}c_s(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jl})\delta(t-t')\times[\delta_{\mathbf{n},\mathbf{n}'}+(1-2\delta_{iz})(1-2\delta_{jz})\delta_{\mathbf{n},\tilde{\mathbf{n}}'}], \tag{13b}$$

where $\tilde{\mathbf{n}}=(n_x, n_y, -n_z)$. Now defining the scaled variables

$$x=-2\sqrt{2}\frac{2\pi}{b\sqrt{R_c}}\tilde{u}_x(\mathbf{n}_1,t), \tag{14a}$$

$$y=2\sqrt{2}\frac{\pi}{\sqrt{bR_c}}\tilde{\theta}(\mathbf{n}_1,t), \tag{14b}$$

$$z=\frac{2\pi}{R_c}\tilde{\theta}(\mathbf{n}_2,t), \tag{14c}$$

with

$$R_c=\frac{(k_c^2+\pi^2)^3}{k_c^2}, \tag{15a}$$

$$b=\frac{(2\pi)^2}{k_c^2+\pi^2}, \tag{15b}$$

$$\tau_1=\frac{1}{k_c^2+\pi^2}, \tag{15c}$$

we obtain the system

$$\tau_1\dot{x}=-\sigma(x-y)+\sqrt{D_x}\xi_x(t), \tag{16a}$$

$$\tau_1\dot{y}=-y+x[\tilde{r}(t)-z]+\sqrt{D_y}\xi_y(t), \tag{16b}$$

$$\tau_1\dot{z}=-b(z-xy)+\sqrt{D_z}\xi_z(t), \tag{16c}$$

which corresponds to Eqs. (2.7) of Ref. [1], but includes the zero-mean Gaussian white noises ξ_x, ξ_y, and ξ_z. Their intensities and correlations are obtained from Eqs. (13) and (15), and are given by

$$\langle \xi_i(t)\xi_j(t')\rangle=\delta_{ij}\delta(t-t'), \tag{17}$$

$$D_x=\frac{\tau_1 k_c^2}{bR_c}c_s, \quad D_y=\frac{\tau_1 k_c^2}{bR_c^2}c_q, \quad D_z=\frac{\tau_1 b k_c^2}{2R_c^2}c_q. \tag{18}$$

The periodic forcing is given by

$$\tilde{r}(t)=\frac{R(t)-2\pi\tilde{S}^{\infty}(2,t)}{R_c}. \tag{19}$$

The conversion to rigid boundary conditions is now performed by taking [1]

$$k_c=3.117, \quad b=2, \quad \tau_1=\frac{1}{2\pi^2}, \quad R_c=1707.8, \tag{20}$$

$$\tilde{r}(t)=1+\epsilon_0+\delta\mathrm{Re}\{f(t)\}, \tag{21}$$

where $\epsilon_0=(T_l-T_u)/R_c$, $\delta=A/R_c$, and

$$f(t)=\frac{9\pi^4\sqrt{i\omega}e^{-i\omega t}}{2\tan(\sqrt{i\omega}/2)(\pi^2-\sqrt{i\omega})(9\pi^2-\sqrt{i\omega})}. \tag{22}$$

The experiment we want to compare with was performed by Meyer, Ahlers, and Cannell [6]. The system was forced to sweep repeatedly through its convective threshold by setting upper and lower plate temperatures like the ones in Eqs. (5) with $\omega=1$. In this experiment an order-disorder transition was observed. This is a sharp transition in the (ϵ_0,δ) plane between "stochastic" behavior (the convective cell pattern is not reproduced for successive cycles) and "deterministic" behavior (the same convective pattern reappears in successive cycles). This transition is depicted in Fig. 1.

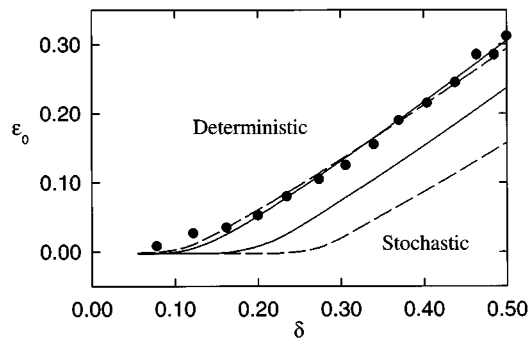

FIG. 1. Order-disorder transition line. The circles are the experimental data of Meyer, Ahlers, and Cannell [6]. The dashed line is the ODTL predicted by the SH equation of Ref. [4] for thermal noise strength (lower line) and for a noise strength 5×10^4 times the thermal one (upper line). The continuous line is the ODTL predicted by the Lorenz model (16) for c_s and c_q, equal to their thermal values (lower line) and for 200 times their thermal values (upper line).

The order-disorder transition line (ODTL) can be analytically defined as the curve on the ϵ_0-δ plane, where the two-times self-correlation of the velocity field after one period of the driving, decays to half its equal-times value (see, e.g., Refs. [2,15]), which for the Lorenz model gives

$$\langle x(t+2\pi/\omega)x(t)\rangle=\tfrac{1}{2}\langle x^2(t)\rangle. \tag{23}$$

Several numeric and (approximate) analytic computations (see, e.g., [2,4,8,15]) using the "standard theory" (SH equation) predict a transition line, for thermal noise intensity, far lower than the experimental data. The prediction of the SH equation for thermal noise [4] is shown in Fig. 1. The ODTL predicted from the "standard theory" can be made to fit the experimental data, only by taking the internal noise strength as an adjustable parameter and setting it to 5×10^4 times its thermal value [2,4], as shown in Fig. 1.

Integrating system (16) of stochastic differential equations by a Runge-Kutta method [16], and averaging over 10^5 realizations, we computed the ODTL predicted by the Lorenz model for thermal noise intensities. The result is plotted in Fig. 1. It can be seen that the prediction of the Lorenz model lies approximately midway between that of the SH equation and the experimental ODTL. We also have taken the noise intensities c_s and c_q as adjustable parameters, and found that the Lorenz model prediction fits the experimental ODTL for values ~200 times the thermal ones, as shown in Fig. 1.

It must be stressed that the Lorenz model reduces to the amplitude equation [1] for the static case and large Prandtl numbers, and the predictions of this equation and that of the SH equation correctly describe the results of static experiments below the convective threshold [12]. Thus it can be expected that for experiments in which the threshold is not crossed, the SH equation would still give an accurate description of the dynamics. This has indeed been verified in recent experiments [12].

On the other hand, for experiments like those in Ref. [6], where the convective threshold is repeatedly and swiftly crossed, the Lorenz model presented here shows a much closer fitting of the ODTL data for thermal noise intensities, and a discrepancy between this noise intensity and the one required to fit the data much lower than that of the ''standard theory.'' Previous models, e.g., the SH equation or the amplitude equation, rely on a single first-order evolution equation for an order parameter or for the amplitude of its more unstable mode, and these equations are thus of a purely dissipative character. The Lorenz model can be recast as a second-order evolution equation for the velocity variable x of Eq. (16a), as has been pointed out by Takeyama [17] [see, e.g., Eq. (2.18) of Ref. [1]], so it includes at least some of the inertial effects present in the Oberbeck-Boussinesq equations. The better results of the Lorenz model for dynamically driven experiments like the one in Ref. [6]—though not yet enough to give an exact matching between theory and experiment—show that these inertial effects should not be neglected *a priori* in nonautonomous dynamic experiments.

The effects of (external) noise in the imposed vertical temperature gradient can be included in the present model in the same way as it was done for the SH equation [14,4]: additive noise in T_l or T_u appears as multiplicative noise in Eq. (16b). However, as it happens for the SH equation, the fitting of the experimental ODTL requires external noise intensities far greater than those compatible with recent experiments [12].

Perhaps the worse-justified approximation made in formulating the Lorenz model is the horizontal wave-number truncation. The model can be improved by retaining the full horizontal wave-number dependence in a vertical eigenfunction expansion of the Oberbeck-Boussinesq equations (1). The lowest nontrivial truncation (retaining the same vertical modes as in the Lorenz model presented here) can be handled with relative ease in the mean-field approximation introduced in Ref. [4]. As shown in Fig. 2, this gives results for the ODTL that are barely distinguishable from those of the Lorenz model (16) and those of *its* mean-field approximation, giving confidence that the Lorenz model results for the ODTL are not fundamentally biased by its poor description of the horizontal wave-number dependence. These results will be presented at length elsewhere.

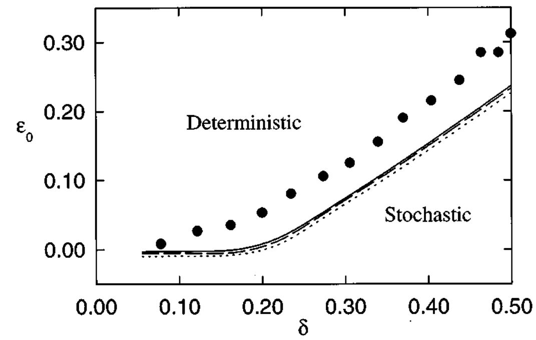

FIG. 2. Order-disorder transition lines predicted by the Lorenz model (16) (continuous line), by its mean-field approximation (dashed line), and by retaining the full horizontal wave-number dependence (dotted line). All curves correspond to thermal noise intensities. The circles are the experimental data of Meyer, Ahlers, and Cannell [6].

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